

Functions and Relations

Every day, we encounter situations where one quantity depends on another. The temperature outside depends on the time of day. The cost of a pizza depends on the number of toppings. The distance a car travels depends on how long it has been driving. In each of these situations, there is a special mathematical relationship connecting two quantities — and understanding that relationship is at the heart of algebra.

In mathematics, we use the word **relation** to describe any pairing between two sets of values. But not every relation behaves in a predictable, organized way. Some relations have a very special property: for every input, there is exactly one output. When a relation has this property, we call it a **function**.

Learning to recognize functions — and to distinguish them from relations that are not functions — is one of the most important skills in eighth-grade mathematics. This lesson will build that understanding step by step, using tables, diagrams, ordered pairs, and graphs.

The Challenge:

A vending machine takes your money and gives you exactly one item per selection. Could a vending machine ever give you two different items for the same button? What would happen to the machine — and to mathematics — if every input could produce more than one output? Think about this as you work through today's lesson.

Understanding the Pattern

To understand what makes a function special, let us start with something familiar: a simple input-output machine.

Imagine a machine where you drop in a number, and the machine processes it and drops out a result. If you put in the number 3 and the machine always gives back 9, that is predictable and reliable. Every time you input 3, you get 9. That consistency is exactly what defines a function.

Now let us look at a concrete example using a table of values.

Table A:

Input (x)	Output (y)
1	5
2	8
3	11
4	14

Study this table carefully. Notice that each input value appears exactly once, and each input is paired with exactly one output. The input 1 always gives 5. The input 2 always gives 8. There is no confusion, no ambiguity. This table **represents a function**.

Table B:

Input (x)	Output (y)
1	5
2	8
2	13
4	14

Now look closely at Table B. The input value 2 appears twice — once paired with 8, and once paired with 13. If you put 2 into this machine, you cannot be sure what you will get out. This unpredictability means Table B **does not represent a function**.

Understanding Functions with Mapping Diagrams

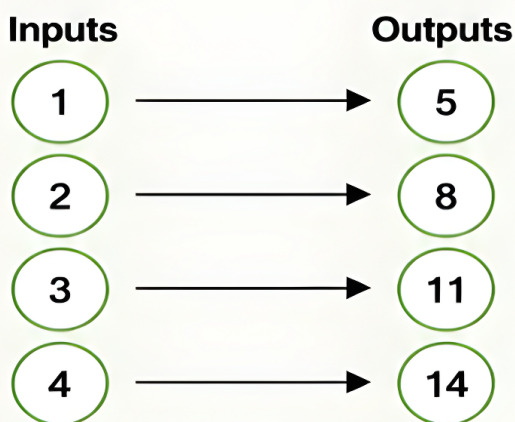
The key idea from those two tables is this: a function never assigns two different outputs to the same input. One input, one output — always.

We can also represent relations using mapping diagrams, sometimes called arrow diagrams. These diagrams show inputs on the left side, outputs on the right side, and arrows connecting each input to its output or outputs.

Definition: Function

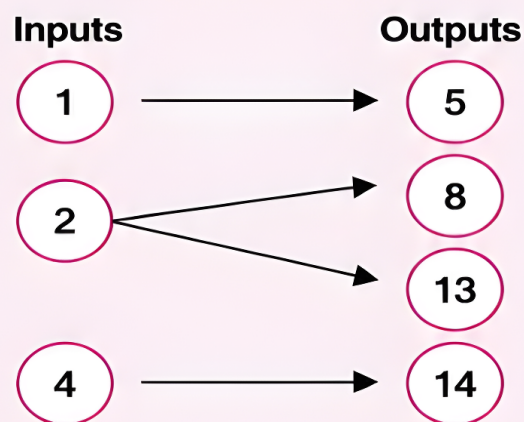
A function is a relation in which each input is assigned exactly one output. No input is assigned to more than one output.

Mapping Diagram A — Function:



Each input has exactly one arrow leaving it, pointing to exactly one output. **This is a function.**

Mapping Diagram B — Not a Function:



The input 2 has two arrows leaving it, pointing to two different outputs. This violates the function rule. **This is not a function.**

Notice something important: in a function, **two different inputs are allowed to share the same output**. For example, inputs 3 and 5 could both map to output 10 — that is perfectly acceptable. The rule only goes one direction: each input must have **one and only one output**.

Recognizing Functions from Ordered Pairs

A relation can also be written as a set of ordered pairs, where each pair has the form (input, output) or (x, y) .

To determine whether a set of ordered pairs represents a function, apply the same rule: check whether any x -value (input) appears more than once with a different y -value (output).

Example — Set 1:

$\{ (2, 7), (4, 11), (6, 15), (8, 19) \}$

Check the x -values: 2, 4, 6, 8.

Each x -value appears exactly once. No input is repeated.

This is a function.

Example — Set 2:

$\{ (3, 9), (5, 25), (3, 12), (7, 49) \}$

Check the x -values: 3, 5, 3, 7.

The value **3** appears twice — once paired with 9, and once with 12. The same input produces two different outputs.

This is not a function.

Example — Set 3:

$\{ (1, 4), (2, 4), (3, 4), (5, 4) \}$

Check the x -values: 1, 2, 3, 5.

Every x -value is different. Even though all four pairs share the same output (4), that does not break the rule.

Each input still has exactly one output.

This is a function.

Key strategy: List all the x -values and look for duplicates. If any x -value repeats with a different y -value, the relation is not a function.

The vertical line test

When a relation is shown as a graph on a coordinate plane, we use a visual tool called the **Vertical Line Test** to determine whether the graph represents a function.

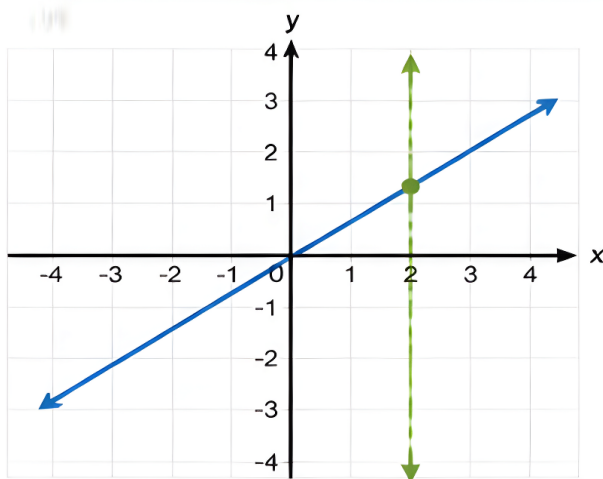
The idea behind this test connects directly to the definition of a function. Each x -value on the horizontal axis represents an input. If a vertical line drawn at any x -value crosses the graph in more than one point, that means the same input is paired with more than one output — and the relation is not a function.

How to apply the Vertical Line Test:

Imagine sliding a vertical line (like a ruler held straight up and down) across the entire graph from left to right.

- If the vertical line **never** touches the graph in more than one point at the same time → the graph **IS** a function.
- If the vertical line **ever** touches the graph in two or more points at once → the graph is **NOT** a function.

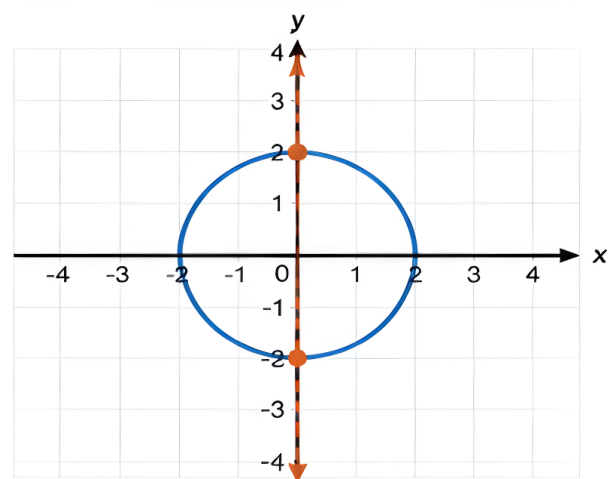
Visual Example 1 — A straight line (function):



Any vertical line drawn anywhere on the graph will intersect it at exactly one point. This graph passes the Vertical Line Test.

It is a function.

Visual Example 2 — A circle (not a function):



A vertical line drawn through the middle of the circle will intersect it at two points — one on the top half and one on the bottom half. This graph fails the Vertical Line Test.

It is not a function.

Understanding relations and functions

In mathematics, we use relations and functions to describe how values are connected. Let's explore what these ideas mean.

Key concepts

Relation

A relation is any set of ordered pairs (x, y) that connects values from one set (inputs) to values from another set (outputs). Relations can be represented as tables, mapping diagrams, ordered pair sets, or graphs on a coordinate plane.

Example: $\{ (1, 3), (2, 5), (4, 9) \}$

Function

A function is a relation in which every input (x) is paired with exactly one output (y) . No input can be assigned to two different outputs.

Rule: Each x -value appears only once as an input, or if it appears more than once, it must always pair with the same output value.

Example: $\{ (1, 4), (2, 7), (3, 10) \}$

✓ **Function**

$\{ (1, 4), (1, 9), (3, 10) \}$

✗ **Not a function**

Writing linear functions

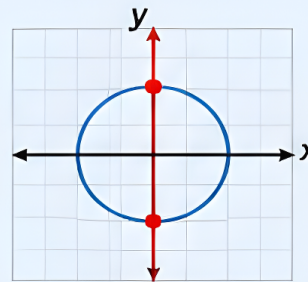
A linear function can be written in different forms, such as slope-intercept form ($y = mx + b$), where m is the slope and b is the y -intercept. We can also write a linear function when given a table, a graph, or a verbal description.

Vertical line test

A graphical method used to determine whether a relation shown on a coordinate plane is a function.

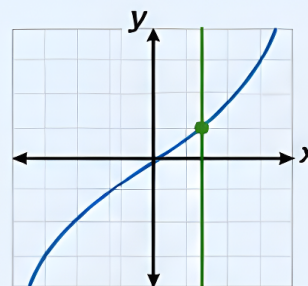
If any vertical line drawn on the graph intersects the graph at more than one point, the graph does **NOT** represent a function.

If every vertical line intersects the graph at exactly one point (or not at all), the graph **DOES** represent a function.



Not a function

The vertical red line intersects the graph at two points.



Function

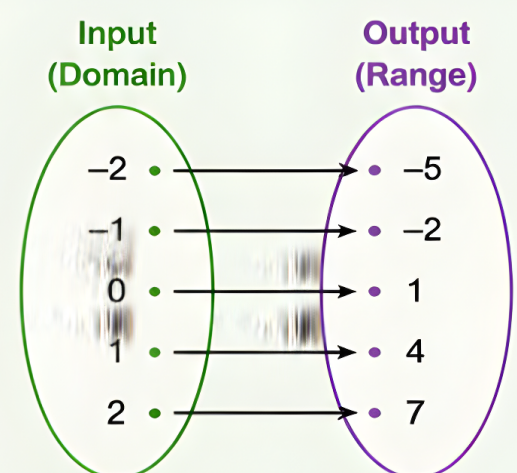
The vertical green line intersects the graph at one point.

Input and output

Input (Domain): The set of all x -values (independent variable) that are used in a relation or function. Also called the domain.

Output (Range): The set of all y -values (dependent variable) that result from the inputs. Also called the range.

In a function, each domain value maps to exactly one range value.



Each input has exactly one output.

Examples Applied

Example 1 — Determining a function from a table

A school cafeteria records the number of lunch trays washed each hour:

Hour (x)	Trays Washed (y)
1	24
2	31
3	28
4	35

Solution: Examine the x-values: 1, 2, 3, 4. Each hour appears exactly once. No input is repeated. Every input has one and only one output.

This table represents a function.

Example 2 — Determining a function from ordered pairs

Determine whether the following set represents a function:

$$\{ (-3, 7), (0, 2), (4, 9), (0, 5), (6, 1) \}$$

Solution: List the x-values: $-3, 0, 4, 0, 6$. The value 0 appears twice — once paired with 2, and once paired with 5. The input 0 maps to two different outputs.

This set does NOT represent a function.

Determining whether a relation is a function

A relation is a set of ordered pairs that shows how inputs are related to outputs. A function is a special type of relation where each input is paired with exactly one output. You can determine whether a relation is a function in different ways. Below are two common methods with examples.

Definition: Function

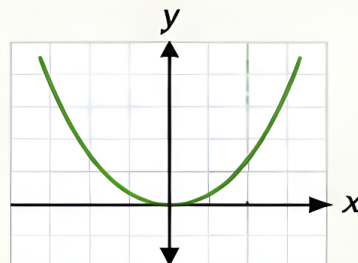
A relation is a function if and only if each input is paired with exactly one output. An input cannot be paired with more than one output.

Inputs	Outputs
1	4
2	5
3	6

This is a function.

Key idea: Vertical line test

A graph represents a function if every vertical line intersects the graph at most once. If any vertical line crosses the graph more than once, then the relation is not a function.



Every vertical line crosses the graph at most once. This is a function.

Example 3 — Mapping diagram analysis

A mapping diagram shows the following connections:

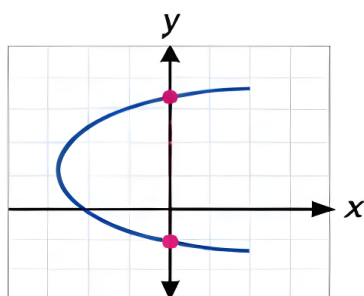
Inputs	Outputs
10	100
20	400
30	900
40	400

Solution: Check each input. The value 10 maps to 100. The value 20 maps to 400. The value 30 maps to 900. The value 40 also maps to 400. Notice that both 20 and 40 share the output 400 — but that is allowed. What matters is that each input has only one arrow. No input maps to two different outputs.

This mapping diagram represents a function.

Example 4 — Applying the Vertical Line Test

A graph shows a curve that starts at the bottom-left of the coordinate plane, curves upward, and loops back to the left, forming a shape that resembles the letter “C” turned sideways.



Solution: Imagine placing a vertical line near the center of the graph. That vertical line crosses the curve in two places — once on the upper portion of the curve and once on the lower portion. This means one x-value is paired with two different y-values.

This graph fails the Vertical Line Test and does NOT represent a function.

Special cases

Can two inputs share the same output?

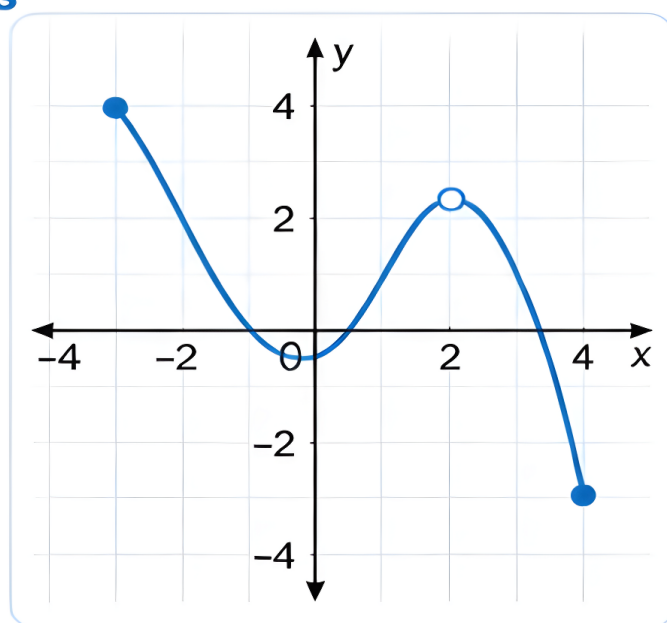
Yes — and this is a common source of confusion. A function allows multiple inputs to map to the same output. For example, the set $\{ (2, 9), (5, 9), (8, 9) \}$ is a perfectly valid function. Every input (2, 5, and 8) maps to exactly one output (9), even though they all share the same output value. The rule only restricts inputs: each input must have one output, but outputs can be shared.

What if an input appears twice with the same output?

Consider the set $\{ (4, 11), (4, 11), (7, 20) \}$. The input 4 appears twice, but both times it maps to the same output, 11. Technically this is a repeated ordered pair, not a violation of the function rule. The input 4 is still only associated with one output value. This is still a function, though the repetition is unnecessary.

Graphs with gaps or open circles

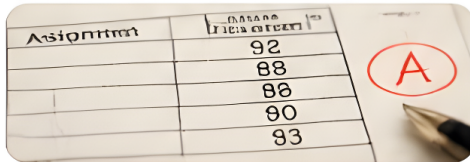
Some graphs have breaks, gaps, or open circles at certain points. When applying the Vertical Line Test, open circles mean that specific point is excluded from the graph. If a vertical line would pass through only an open circle and no solid point, that x -value does not create a conflict. Always distinguish between open (excluded) and filled (included) points when reading graphs carefully.



Synthesis and Application

Functions are one of the foundational ideas in all of mathematics. The concept of a reliable, predictable input-output relationship appears in science, technology, economics, and engineering. When we say that a relationship is a function, we are making a powerful statement: given any input, we can always know exactly what output to expect.

Where functions appear in real life:



Assignment	Points	Grade
	92	A
	88	
	88	
	90	
	93	

A student's final grade depends on their total points — each point total corresponds to exactly one grade.



A car's fuel gauge reads one specific level for each amount of gasoline remaining in the tank.



A recipe produces one specific yield for each exact set of ingredient quantities.



A ZIP code corresponds to one specific geographic region — no ZIP code belongs to two different locations simultaneously.

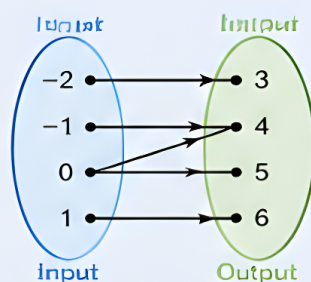
Helpful strategies to remember:

When working with **tables or ordered pairs**, look for repeated x-values. If the same x-value appears with two different y-values, it is not a function.

x	y
-2	3
-1	4
1	5
1	7

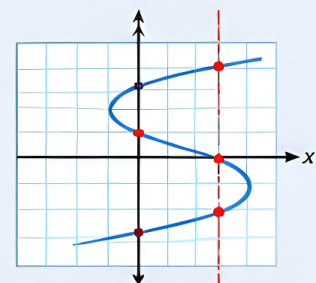
Not a function ($x = 1$ has two different y-values)

When working with **mapping diagrams**, look for any input that has more than one arrow leaving it.



Not a function (input 0 has two arrows)

When working with **graphs**, apply the **Vertical Line Test** — slide a vertical line across the entire graph and check for double intersections.



Not a function (vertical line intersects twice)

Key takeaway

Understanding functions prepares students for deeper work with linear equations, slope, graphing, and eventually advanced topics like quadratic and exponential functions. Every time you encounter a graph or equation this year, the first question to ask is:

is this a function?